AN UPPER BOUND ON THE CRITICAL DENSITY FOR ACTIVATED RANDOM WALKS ON EUCLIDEAN LATTICES

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ABSTRACT. We show the critical density for activated random walks on Euclidean lattices is at most one.

1. INTRODUCTION

Given a graph, the activated random walks (ARW) model starts with an initial configuration in which each vertex is occupied by a finite number of sleeping or active particles. Beginning with this initial configuration, each active particle performs an independent, rate one, random walk, while sleeping particles stay put. If a sleeping particle occupies the same vertex as an active particle, it becomes active immediately. Finally, active particles fall asleep independently at a rate $\lambda > 0$. We examine this model on $\mathbb{Z}^d$, where we suppose that initially, each vertex contains an i.i.d. Poisson number of active particles with expected value μ. The Poisson distribution plays no special role and could be replace by other distributions.

One obvious question on the long term behavior of the system, is whether or not we have fixation, which by translation invariance is equivalent to whether the number of active particles that visit the origin is finite almost surely. In Theorem 2.1 below, we show that for $\mu > 1$, we almost surely do not have fixation.

For this, we rely on the technical framework developed in [1]. In this paper, the existence of the process is proved, and it is shown that the probability of finitely describable events can be approximated by finite systems. Let $\mathbb{P}^\mu$ be the probability measure on the model described above, and let $\mathbb{P}^\mu_M$ be the measure on the model with all particles outside of B_M removed, where $B_M = \{x \in \mathbb{Z}^d : \|x\| \leq M\}$. If A is an event measurable with respect to what happens in some finite subset up to some time $t < \infty$,

$$\mathbb{P}^\mu(A) = \lim_{M\to\infty} \mathbb{P}^\mu_M(A).$$

The second tool from [1], is a graphical representation for systems with finitely many particles that has the desirable properties of monotonicity and commutativity of certain parameters. Here is a loose description. Let there be a universal clock that will ring with the appropriate rate, fix some label for each particle, and let there be an i.i.d. sequence of labels, independent of the clock. Also, at each site, let there be an i.i.d. sequence of envelopes, each one containing some instruction to be performed. When the clock rings for the first time, it will ring for the particle indicated by the first label in the sequence and at that moment this particle will perform some action. If the particle is sleeping, nothing happens. If the particle is active, it will open the first envelope at that site, burn the envelope and perform the action written inside. The instruction may be to jump to a specific neighbor,

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or to try to sleep. Thus there are two types of envelopes: jump envelopes and sleep envelopes. If the particle tries to sleep but there are other particles on the same site, the envelope is burned anyway. This representation has the activated random walk process as a natural marginal, and is described formally for one dimension in the above reference.

The commutativity property says the following. Suppose for a given realization of the process, the system fixates, that is, all particles are passive for all large enough times (of course starting with finitely many particles this happens a.s.). Then by changing the label sequence and the universal clock, the system will stabilize at exactly the same state, except that some particles may be permuted. Furthermore, the amount of envelopes burned at each site is also preserved. So the final state of the system is determined by the initial conditions (positions and types of the particles) and the by sequences of envelopes. The second property, monotonicity, states the following. Suppose for some realization of the envelopes and initial conditions there is fixation. Take a new configuration by deleting some particles on the original one, changing the type of some particles from active to sleeping and inserting some sleep envelopes at some sites' envelope sequences. Then for this new configuration the system also stabilizes and the final number of envelopes that are burned at each site (not counting the ones inserted) does not increase.

This framework was used to show that there is at most one phase transition in the model for Euclidean lattices and that in one dimension, there exists a phase transition for some $0 < \mu_c \leq 1$.

We utilize these properties to modify the finite approximations of the process to "sleepier" ones, and show that when the density μ is higher than one, the number of visits to the origin still goes to infinity almost surely, hence there is no fixation.

2. Result

Here we settle one of the open problems posed in the concluding remarks of the [1], and prove that for Euclidean lattices of all dimensions, $\mu_c \leq 1$. This is shown by using the below theorem in conjunction with Theorem 1.1 in [1].

Theorem 2.1. *Consider the activated random walk model on $\mathbb{Z}^d$. Then for any $\mu > 1$ the system does not fixate.*

Proof. For $r \in \mathbb{N}$, let A_r be the event that the origin is visited by an active particle at least r times before fixation. Fixing d, and $\mu > 1$, we prove the above by showing that for all $r \in \mathbb{N}$,

$$\lim_{M\to\infty} \mathbb{P}_M^{\mu}\left(A_r\right) = 1. \tag{2.1}$$

Let $n = n(M, \omega)$ be the number of particles in the system. Since $\mu > 1$, for some $\epsilon = \epsilon(\mu)$,

$$\lim_{M\to\infty} \mathbb{P}\left[n > (1+\epsilon)\left|B_M\right|\right] = 1. \tag{2.2}$$

Let $F = F(M)$ be the event in (2.2) that happens with high probability.

For any finite M, we can use the monotonicity and commutativity proven in [1] to reduce the model to the following IDLA-like process.

First notice that regardless of λ (the rate by which particles fall asleep), if two particles or more occupy the same site, they will all be active almost surely. Considering one particle, we are thus assured it will continue its random walk at least

until it has reach an unoccupied site. Conditioning on F, we fix some order on $N = \lceil (1+\epsilon)|B_M| \rceil$ randomly chosen particles, and by adding sleep envelopes, we make the remaining $n - N$ particles static so they won't interfere. We modify the label sequence so that each of the N particles in turn begins a random walk from its initial location until reaching a site unoccupied by other particles (possibly its starting location - in which case it wouldn't move at all). Once reaching such a site, we force the particle to remain there forever by inserting a sleep envelope for each of its movement attempts . Let $\mathcal{P}$ denote the probability measure on this "embedded" Markov process. Let V be the number of particles that visit zero before stopping.

By the monotonicity, to prove (2.1) it suffices to show that V grows linearly with M with high probability. Formally,

$$\mathcal{P}\left[V > \frac{\epsilon}{4}M \;\middle|\; F(M)\right] \to 1 \tag{2.3}$$

To prove (2.3) we use an idea from the original IDLA paper [2], which is even simpler to apply in our setting.

Let $\{X_i\}_{i=1,\ldots,N}$ be the starting locations of the N particles. Since the density is i.i.d., for any $x \in B_M, \mathbb{P}^{\mu}_M[X_i = x] = |B_M|^{-1}$. Unlike the real model, we let the walks continue forever, but mark (e.g. by coloring) the locations where they first visit an unmarked vertex. We start with all vertices unmarked. Let each walk run in turn (without stopping) and mark the first unmarked vertex it visits. This may be the initial placement of the particle (and will be in most cases). We call the component of marked vertices at each step the cluster.

Let W be the number of walks that visit 0 before exiting B_M.

Let L be the number of walks that visit 0 before exiting B_M, but after leaving the cluster (i.e. after stopping in the original model).

Note that $W - L$ is the number of visits to zero of particles that haven't left the cluster, and is at most equal to V. Thus we have

$$\begin{aligned} \mathcal{P}[V < \frac{\epsilon}{4}M] &< \mathcal{P}[W - L < \frac{\epsilon}{4}M] \\ &\le \mathcal{P}[W - \frac{\epsilon}{4}M \le a] + \mathcal{P}[L \ge a] \end{aligned}$$

for any real a. We choose $a = (1 - \frac{\epsilon}{2})E[W]$. We bound the above terms by calculating the expected value of M and L. Let τ_0 be the first hitting time of 0 of a random walk, and let τ_M be the first exit time from B_M.

$$E[W] = \sum_{i=1}^{N} \sum_{x \in B_M} \mathcal{P}_x[\tau_0 < \tau_M] P(X_i = x) = \frac{N}{|B_M|} \sum_{x \in B_M} \mathcal{P}_x[\tau_0 < \tau_M]$$

$E[L]$ is hard to calculate, but note that each walk that contributes to L can be tied to the unique point at which it exits the cluster. Thus, by the Markov property, if we start a random walk from each vertex in B_M and let $\hat{L}$ be be the number of such walks that hit 0 before exiting B_M, we have $\mathcal{P}[L \ge a] \le \mathcal{P}[\hat{L} \ge a]$.

Thus, $E[\hat{L}] \ge E[L]$, and we have

$$E[\hat{L}] = \sum_{x \in B_M} \mathcal{P}_x[\tau_0 < \tau_M]$$

So we have $E[W] \geq (1+\epsilon)E[\hat{L}]$ which gives us for $\epsilon < \frac{1}{2}$

$$\mathcal{P}[L \geq a] \leq \mathcal{P}[\hat{L} \geq (1-\frac{\epsilon}{2})(1+\epsilon)E[\hat{L}]] \leq \mathcal{P}[\hat{L} \geq (1+\frac{\epsilon}{4})E[\hat{L}]]$$

We can lowerbound $E[\hat{L}]$ by using the Green function identity:

$$\mathcal{P}_x[\tau_0 < \tau_M] = \frac{G_M(x,0)}{G_M(0,0)}$$

where $G_M(a,b)$ is the average number of visits of a random starting at a to b before leaving B_M. By symmetry of the Green function we can write

$$E[\hat{L}] = \sum_{x \in B_M} \mathcal{P}_x[\tau_0 < \tau_M] = G_M(0,0)^{-1} \sum_{x \in B_M} G_M(0,x) = G_M(0,0)^{-1} E_0[\tau_M].$$

By the optional stopping theorem with the martingale $\|X(t)\|^2 - t$ we have $E_0[\tau_M] = M^2$. Second, $G_M(0,0) = M$ for the line and is smaller for higher dimensions (e.g. by the monotonicity law for electric networks).

Since $E[W] \geq E[\hat{L}] \geq M$, we have $\mathcal{P}[W - \frac{\epsilon}{4}M \leq a] \leq \mathcal{P}[W \leq (1-\frac{\epsilon}{4})E[W]]$.

Since $\hat{L}$ and W are both sums of indicators, we can use standard concentration inequalities, and the lower bound on $E[\hat{L}]$, to show exponential decay in N of $\mathcal{P}[V < \frac{\epsilon}{4}M]$ which proves (2.3), and we are done. $\square$

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References

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[2] Gregory F. Lawler, Maury Bramson, David Griffeath. Internal Diffusion Limited Aggregation, *The Annals of probability* (1992) Vol. 20, No. 4, 2117-2140